Præcipuarum Functionum Trigonometricarum per Analysin Infinitorum Explicatio.

Dissertatio,

Quam,

Consentiente Ampl. Facultate Philosophica

Præside

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Astron. Prof. Reg. et Ord.,

pro Gradu obtinendo

publice proponit

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Scanus.

In Academia Carolina die XXI Junii MDCCCXLI.

Part. II.

LUNDÆ,

typis excudit CAROLUS FR. BERLING,

Universitatis Typographus,

MDCCCXLI.

Quo igitur majori cum fructu problemate 12:mo in hac quæstione utamur, ponamus $\frac{\pi}{4}=a+b$, et tangens $a=\frac{1}{2}$; quo fiet $tang\, b=tang\left(\frac{\pi}{4}-a\right),=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3}$.

Ope igitur problematis hujus prodibit

$$a=\tfrac{1}{2}-\frac{1}{3.2^3}+\frac{1}{5.2^5}-\frac{1}{7.2^7}+\frac{1}{9.2^9};-\ldots.$$

$$b=\tfrac{1}{3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\frac{1}{7.3^7}+\frac{1}{9.3^9}-\ldots.;\text{ atque hinc}$$

$$\frac{\pi}{4}=a+b=\left(\frac{3+2}{6}\right)-\tfrac{1}{3}\left(\frac{3^3+2^3}{6^3}\right)+\tfrac{1}{5}\left(\frac{3^5+2^5}{6^5}\right)-\tfrac{1}{7}\left(\frac{3^7+2^7}{6^7}\right)+\ldots.^{*)}$$

Si cum Cl. Machin ponamus $\frac{\pi}{4}=4a-b$ atque $tang\, a=\frac{1}{5}$, erit tangens $4a$

$$=\frac{4\, tang\, a-4\, tang^3 a}{1-6 tang^2 a+tang^4 a}=\frac{4\left(\frac{1}{5}-\frac{1}{125}\right)}{1-\frac{6}{25}+\frac{1}{625}}=\frac{4(125-5)}{625-150+1}=4\left(\frac{120}{476}\right)$$

$=\frac{120}{119}$; unde $tang\,(4\,a-\pi)=\frac{1}{239}=tang\ b$. Hinc

$$4\,a=4\left(\frac{1}{5}-\frac{1}{3.5^3}+\frac{1}{5.5^5}-\frac{1}{7.5^7}+\frac{1}{9.5^9}-\ldots\right)=A,$$

$$b=\frac{1}{239}-\frac{1}{3(239)^3}+\frac{1}{5(239)^5}-\ldots\ldots=B,\text{ atque}$$

$$\frac{\pi}{4}=A-B.$$

Seriem hanc valde convergentem esse ut nemo non viderit, ita quisque facile intelliget plures ejusmodi, et quidem illa magis convergentes, justo adhibito delectu tangentis a, hac methodo posse inveniri. **)

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*) In eligendo valore tangentis $a=\frac{1}{2}$, secuti sumus exemplum in *Lehrbuch des höhern Kalkuls* von Lubbe. Berlin 1825 pag. 44, ut admoneremus simul de erroneo signo in numeratoribus, ibidem occurrente, nec in corrigendorum indicem postea relato.

**) Quousque in hac re humana diligentia satis, ut videtur, superque processerit, testatur valor num. π (v. c. in Tabulis *Cl. Vega*) usque ad 140:mam notam decimalem

Schol. Quomodo exponens rationis peripheriæ ad diametrum majoribus numeris vel, quod eodem redit, pluribus notis decimalibus exhibitus ad fractiones simplicioris formæ et tamen, pro simplicitate sua a vero minimum aberrantes reducatur, vel, versa vice, a simpliciore ad magis compositam veroque propiorem expressionem evehatur, eleganter docet theoria fractionum continuarum.

Invento π facile obtinebitur $sin 1''$ e formula $b = \frac{r\pi g}{180^0} = \frac{\pi}{180.60^2}$;

r et g, suo utroque in genere, unitati æqualibus.

Probl. 15. *Dato angulo alterum invenire, trigonometrica functione ex illo pendentem.*

In problematis præcedentibus exponens differentialis $\frac{du}{dx}$ variabiles heterogeneas solas respexit. Nihil autem impedit, quo minus homogeneæ esse possint; quo casu id etiam obtinetur, ut, quatenus ipsæ inter se computando comparentur, neutro factorum illorum, $sin 1''$ et $\frac{1}{sin 1''}$, sit opus. Inter ea, quæ, hoc nomine, artem analyticam provocant problemata, celebre est illud Kepleri hac æquatione propositum: $u = z + e' \sin u$, ubi u anomalia excentrica, z anomalia media, e' in partes arcus reducta excentricitas e, posito semiaxe majore $= 1$. In qua æquatione si, data u, quæratur z, res non multum habebit negotii.

Erit enim $z = u - e' \sin u$, ubi $\sin u$ vel e tabellis desumi potest, vel, si computare juvat, e formula ea, qua et illæ constructæ sunt. Quam quidem formulam jam ob alteram Kepleriani problematis partem repetamus, scribendo

$$z = (1 - e')u + e'\left(\frac{u^3}{1.2.3} - \frac{u^5}{1.2.3.4.5} + \frac{u^7}{1.2.3.4.5.6.7} - \dots\right);$$

ubi si u per methodum *Reversionis Serierum*, vel aliam, directe progressam, determinanda sit, multis et operosioribus, præsertim in excentricitate paullo majori, terminis computandis defatigabimur. In valde autem parvis foci a centro

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hodie expressus. Computandi illius formula, in Tomo 1. Tabl. Log. Trig, Lipsiæ 1797 ipsi valori præmissa, ut operosior, ita adcuratior, cum Cl. Vega, ejusdem ope, errorem in nota decimali 113, diu latentem, in lucem perspicacissimus protraxerit.

distantiis, quales veterum fere planetarum orbitæ habent, nec ubi omnia scrupulosissime inquirenda sunt, directa illa via haud ita videatur incommoda.

Ut hoc probemus exemplo, neque ab illis, quas attulimus, præceptis alieno, et, quanta in hoc genere liceat, brevitate insuper commendabili, methodum a Cl. *Bossut* in *Traités de Calcul Differentiel et de Calcul Integral,* *) propositam magis, quam explicitam, ita exponemus, ut et omnia uberius tractentur, et quæsitis exponentibus differentialibus comparando ostendatur, quænam illorum adhibendorum ratio sit ad propositum adsequendum commodissima.

In problemate igitur Kepleriano cum data anomalia media non tam proprie excentrica quæratur, quam vera v, ad quam per excentricam, e media, fit transitus, revocemus e theoria orbitæ ellipticæ has, pro quadrante primo, e perihelio computando, adornatas formulas:

$$z = u - e' \sin u \,;\quad \sqrt{\frac{1-e}{1+e}}\, \tang \tfrac{1}{2} v = \tang \tfrac{1}{2} u$$

$$\text{rad. vector } r = 1 - e \cos u = \frac{1-e^2}{1+e\cos v}\,;\quad \cos \tfrac{1}{2} u = \cos \tfrac{1}{2} v \sqrt{\frac{r}{1-e}}\,;$$

Habebimus jam $du = \dfrac{(1-e)^{\frac{1}{2}} \cos^2 \frac{1}{2} u}{(1+e)^{\frac{1}{2}} \cos^2 \frac{1}{2} v}\, dv = \dfrac{r}{(1-e^2)^{\frac{1}{2}}}\, dv$, atque

$dz = du(1 - e \cos u)$, dum factor $\dfrac{1}{\sin 1''}$ in e' supprimitur,

$$= du \times r = \sqrt{\frac{1}{1-e^2}} \times \frac{(1-e^2)^2}{(1+e\cos v)^2}\, dv = \frac{(1-e^2)^{\frac{3}{2}}}{(1+e\cos v)^2}\, dv.$$

Sit dz, ut vult natura rei, constans, excentricitasque e tam parva, ut dignitates ejus, tertiam excedentes, haud multo errore, cifræ æquentur, erit

$$\frac{dv}{dz} = \frac{(1+e\cos v)^2}{(1-e^2)^{\frac{3}{2}}} = (1+e\cos v)^2 (1-e^2)^{-\frac{3}{2}}$$

$$= (1 + 2e\cos v + e^2\cos^2 v)(1 + \tfrac{3}{2} e^2) = 1 + 2e\cos v$$

$$+ e^2 \cos^2 v + \tfrac{3}{2} e^2 + 3e^3 \cos v = 1 + 2e^2 + (2e + 3e^3)\cos v + \frac{e^2 \cos 2v}{2} = Q.$$

$$\frac{d^2 v}{dz^2} = (-2e\sin v - 3e^3 \sin v - e^2 \sin 2v)Q = -2e \sin v - 7e^3 \sin v - e^2 \sin 2v$$

$$- 4e^2 \sin v \cos v - 2e^3 \cos v \sin 2v - e^3 \sin v \cos 2v = -2e\sin v - 7e^3 \sin v$$

*) A Paris, An VI, Tome Premier, pag. 440.

$$-3e^2 sin2v - e^3 sin3v - \tfrac{1}{2} e^3 sin3v -- \tfrac{1}{2} e^3 sin\, v = -2\, e\, sin\, v - \frac{15}{2} e^3 sin\, v$$

$$-3e^2 sin2v - \tfrac{3}{2} e^3 sin3v = Q^{I};$$

$$\frac{d^3v}{dz^3} = \left(-2\, e\, cos\, v - \frac{15}{2}\, e^3 cos\, v - 6e^2 cos\, 2\, v - \frac{9}{2} e^3 cos\, 3\, v\right) Q^{I} = -2\, e\, cos\, v$$

$$-\frac{23}{2}\, e^3 cos\, v - 6e^2 cos\, 2\, v - \tfrac{9}{2}\, e^3\, cos\, 3v - 4\, e^2 cos^2 v - 13\, e^3 cos\, v\, cos\, 2v$$

$$= -2e\, cos\, v - \frac{23}{2}\, e^3 cos\, v - 8e^2 cos2v - \tfrac{9}{2} e^3 cos3v - 2e^2 - \frac{13}{2} e^3 cos\, v - \frac{13}{2} e^3 cos3v$$

$$= -2e^2 - 2e\, cos\, v - 18e^3 cos\, v - 8e^2 cos2v - 11e^3 cos3v = Q^{II};$$

$$\frac{d^4 v}{d z^4} = (2\, e\, sin\, v + 18\, e^3 sin\, v + 16\, e^2 sin\, 2\, v + 33\, e^3 sin\, 3\, v)\; Q = 2\, e\, sin\, v$$

$$+ 22\, e^3 sin\, v + 16e^2 sin^2 v + 33e^3 sin3v + 4e^2 sin\, v\, cos\, v + 32e^3 cos\, v\, sin2\, v$$

$$+ e^3 sin\, v\, cos\, 2\, v = 2\, e\, sin\, v + 22\, e^3 sin\, v + 18\, e^2 sin\, 2\, v + 34\, e^3 sin\, 3\, v$$

$$+ \frac{31}{2} e^3 sin3v + \frac{31}{2} e^3 sinv = 2e\, sin\, v + \frac{75}{2} e^3 sinv + 18e^2 sin2v + \frac{99}{2} e^3 sin3v = Q^{III};$$

$$\frac{d^5 v}{d z^5} = \left(2\, e\, cos\, v + \frac{75}{2} e^3 cos\, v + 36e^2 cos\, 2\, v + \frac{297}{2}\, e^3\, cos\, 3\, v\right) Q = 2\, e\, cos\, v$$

$$+ \frac{83}{2}\, e^3 cos\, v + 36e^2 cos2v + \frac{297}{2} e^3 cos\, 3v + 4e^2 cos\, v^2 + 73\, e^3\, cos\, v\, cos\, 2\, v$$

$$= 2e\, cosv + \frac{83}{2} e^3 cos\, v + 38e^2 cos2v + \frac{297}{2}\, e^3 cos3v + 2e^2 + \frac{73}{2} e^3 cos3v + \frac{73}{2} e^3 cosv$$

$$= 2e^2 + 2e\, cos\, v + 78e^3 cos\, v + 38e^2 cos2v + 185e^3 cos3v = Q^{IV} \text{ cet.}$$

Si quæ ita inventa sunt, theorematì Tayloriano adplicarentur, prodiret

$$v = f\, z = (z+y) = Z + Qy + \frac{Q^{I} x y^2}{2} + \frac{Q^{II} y^3}{1.2.3} + \frac{Q^{III} y^4}{1.2.3.4} + \frac{Q^{IV} y^5}{1.2.3.4.5} + \ldots;$$

sive posito $z = o$, et deinde pro y scripto z,

$$v = z + \left(\frac{2e + \frac{5}{2} e^2 + 3\, e^3}{1}\right) z - \left(\frac{2e + 10e^2 + 29e^3}{1.2.3}\right)^{z^3} +$$

$$+ \left(\frac{2e + 40\, e^2 + 263\, e^3}{1.2.3.4.5}\right)^{z^5} - \ldots;$$ quæ series cum et tardius convergat et valori, calculo subducendo, minus sit commoda, de aptiore jam videndum.

Sit $v = z + A \sin z + B \sin 2Z + C \sin 3Z + \ldots$ erit

$$\frac{dv}{dz} = 1 + A \cos z + 2B \cos 2Z + 3C \cos 3Z + \ldots$$

$$\frac{d^2v}{dx^2} = - A \sin z - 4B \sin 2K - 9C \sin 3Z + \ldots$$

$$\frac{d^3v}{dz^3} = - A \cos z - 8B \cos 2z - 27C \cos 3Z + \ldots$$

$$\frac{d^4v}{dz^4} = A \sin z + 16B \sin 2Z - 81C \sin 3Z + \ldots$$

$$\frac{d^5v}{dx^5} = A \cos z + 32 \operatorname{Cos} 2Z + 243 C \cos 3Z + \ldots$$

Prioribns igitur exponentibus differentialibus cum his respective comparatis, *veraque* anomalia *mediam* in apsidibus æquante, nonnullis permutationibus obtinebitur:

$A = 2e - \frac{e^3}{4}$; $B = \frac{5e^2}{4}$; $C = \frac{13e^3}{12}$; quibus efficietur

$$v = z + \left(2e - \frac{e^3}{4}\right) \sin z + \frac{5e^2 \sin 2z}{4} + \frac{13e^3 \sin 3z}{12} + \ldots$$

Patet igitur in hoc, similibusque problematis, sumtis exponentibus differentialibus, per methodum coefficientium indeterminatorum, functione variabilis scite delecta, aptius celeriusque, quam per theorema Taylorianum, ad optatum exitum perveniri.

Schol. In orbitis excentricitatis majoris, quemadmodum jam diximus, si directa via tentanda est, ut adcuratiores, ita longo terminorum tractu molestiores formulas problema Keplerianum requiret. Plena sunt horum præceptorum locupletiora in hac arte doctissimorum virorum opera. Cum vero, præter necessitatem nemo fatigari velit, multo jure a computantibus antefertur indirecta illa methodus, *Gaussiana* præsertim arte concinnata.

Sit $u = u' + x = z + e' \sin(u' + x)$, ubi u' significat anomaliam excentricam, quomodocumque, prope cognitam, et x vel excessum, vel defectum, quo hæc a vera u differat; habebimus: $u = u' + x = z + e' \sin u' + e' \cos u' x$; vel, si ad signa diversorum quadrantum respiciatur, $u = u' + x = x \pm e' \sin u' \pm e' \cos u' x$.

Porro:

$$e' \cos u = \frac{\log \sin (u'+s) - \log \sin u'}{\log (e' \sin u' + s) - \log e' \sin u'} = \frac{a}{b};$$

designante s minutum secundum, sive unitatem numeri x; atque hinc

$$x = \frac{b}{b \mp a} \left(z \pm e' \sin u' - u'\right).$$

Quæ hoc modo invenietur excentrica u, credita nondum esse plane perfecteque vero valori consentanea, vocetur u'', ut sit iterum $u = u'' + x' = z \pm e' \cos u'' x'$.

Formula u, hoc modo bis calculata, vel in maxima excentricitate planetica v. c. Junonis ($= 0{,}2554996$) valorem exhibebit vix in millenis partibus minuti secundi a vero discrepantem *).

Schol. 2. Quam in hoc problemate solvendo adhibuimus methodum coefficientium indeterminatorum, hæc eandem in præcedentibus operam præstare potuisset. Etenim illa a Tayloriana hac re potissimum differt, quod seriem pro tempore et occasione aptissimam eligat, cum hæc certa et constanti lege dignitatum incrementi nitatur.

Tayloriana tamen hoc præ se fert commodi, ut exponentes differentiales semel tantum determinare opus sit, cum altera determinatio, jam in ipsa formulæ constructione facta, numeratoribus (1); (1.2); (1.2.3); (1.2.3.4); cet. recondita, servetur. Quod ut melius illucescat, sit in exemplo Kepleriano hæc lex coefficientium indeterminatorum

$v = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \dots$; erit

$$\frac{dv}{dz} = A + 2Bz + 3Cz^2 + 4Dz^3 + 5Ez^4 + \dots$$

$$\frac{d^2v}{dz^2} = 2B + 6Cz + 12Dz^2 + 20Ez^3 + \dots$$

*) Theoria motus corporum coetestim in Sectionibus conicis solem ambientium. Auctore C. F. Gauss. Hamburgi 1809, pag. 10. Uberiorem hujus Methodi explicationem dedimus in Dissertatione de Anomalia vera indirecte invenienda, Resp. *Carolo J. Petersson.* Lundæ MDCCCXVIII.

$$\frac{d^3 v}{d z^3} = 6 C + 24 D z + 60 E z^2 + \ldots.$$

$$\frac{d^4 v}{d z^4} = 24 D + 120 E z + \ldots.$$

$$\frac{d^5 v}{d z^5} = 120 E + \ldots. ;$$

qui exponentes, cum prioribus, sive iis, qui ex ipsa problematis natura communiter petuntur, collati, posito v, et cum illo $z, = o$, dabunt:

$$A = 1 + 2 e + \frac{5 e^2}{2} + 3 e^3 ; \; B = o$$

$$C = \frac{-(25 + 10 e^2 + 29 e^3)}{1.2.3} ; \; D = o$$

$$E = \frac{(2 e + 40 e^2 + 263 e^3)}{1.2.3.4.5} ; \text{ cet. Hinc}$$

$$v = z + \frac{\left(2e + \frac{5e^2}{2} + 3e^3\right)}{1} z - \left(\frac{2e + 10e^2 + 29e^3}{1.2.3}\right) z^3 + \left(\frac{2e + 40e^2 + 263e^3}{1.2.3.4.5}\right) z^5$$

quem valorem, iisdem conditionibus, e theoremate Tayloriano jam eruimus.

Eadem exsistet utriusque methodi per alteram adprobatio, si, ad superiores quæstiones regressi, vel lineas trigonometricas ex arcu, vel hunc ex illis ope methodi coefficientium indeterminatorum investigaverimns. Denique per methodum Reversionis Serierum, quam et illa genere suo complectitur, ex quocumque, utralibet methodo jam invento, ad id, quod datum fuit, reverti posse, cum per se satis eluceat, tum uno exemplo jam ostendisse sufficiet.

Scholion 3. Calculum Integralem etiam in his disquisitionibus adhiberi notum est. Attamen si, ut exorsi sumus, ex arcu linea, v. c. sinus, quærendus est, per simplicem et directam, ut patet, integrationem formulæ $du = \frac{dx}{\sqrt{1-x^2}}$ id fieri non licebit. Quærendus scilicet per illam primum arcus u e sinu x, deinde per reversionem seriei e sinu x arcus u. Directe ex arcu in-

venientur sinus, cosinus tangens cet. per formulam Taylorianam, methodum coefficientium indeterminatorum, aliasque ad calculum differentialem proprie pertinentes. Latiore tamen sensu hæc præcepta etiam integrandi dici possint. Est enim talis evolutio integratio indirecta, seu per differentiationem regressiva. Nam si $x = Au + Bu^2 + Cu^3 + Du^4 + Eu^5 + \ldots$ quid aliud innuit, quam quod $\int dx = Au + Bu^2 + Cu^3 + Dx^4 + Ex^5 + \ldots = \int (1-x^2)^{\frac{1}{2}} du$; quæ integratio perficietur, si coefficientes indeterminatos monstrata via quæsiveris. Erit nempe $dx = d(Au + Bu^2 + Cu^3 + Du^4 + \ldots) = du \times Q = (1-x^2)^{\frac{1}{2}} du$;

et hinc $\frac{du}{dx} = \frac{1}{(1-x^2)^{\frac{1}{2}}}$; itemque $\frac{du}{dx} = \frac{1}{Q}$;

e quibus si utrimque, solito ordine, ulteriores exponentes differentiales sumantur, facta comparatione horum, coefficientes emergent integrationem confecturi. Eadem ratio est theorematis Tayloriani, Maclauriniana conditione restricti. Nempe

$$x = \int dx = u + \frac{du}{dx} u + \frac{d^2u}{1.2.dx^2} u^2 + \frac{d^2u}{1.2.3.dx^3} u^3 + \frac{d^4u}{1.2.3.4dx^4} + = \int (1-x^2)^{\frac{1}{2}} du;$$

ubi coefficientes differentiales $\frac{du}{dx}$; $\frac{d^2u}{dx^2}$; $\frac{d^3u}{dx^3}$; cet, ut per formulam illam: $\frac{du}{dx} = \frac{1}{(1-x^2)^{\frac{1}{2}}}$, investigati fuerint, tota erit, illis adhibendis, integratio perfecta.

Inseritur igitur a multis explicatio harum methodorum expositioni Calc. Integralis.

Si vero arcus e linea trigonometrica, v. c. e sinu per integrationem quæretur, per solita integrandi principia habebitur:

$$\int du = \int \frac{dx}{(1-x^2)^{\frac{1}{2}}} = \int dx \left(\frac{1+x^2}{2} + \frac{3x^4}{2.4} + \frac{3.5x^6}{2.4.6} + \frac{3.5.7x^8}{2.4.6.8} \right)$$

$$= x + \frac{x^3}{2.3} + \frac{3x^5}{2.4.5} + \frac{3.5x^7}{2.4.6.7} + \frac{3.5.7x^9}{2.4.6.8.9} + \ldots C$$

ubi cum Constans $C = o$ necesse sit, valor arcus, sinu suo expressi, idem erit, qui supra Tayloriano theoremate inventus est.

Pari modo habebimus arcum u, e cosinu suo x, per integrationem formulæ

$$u = -\frac{dx}{(1-x^2)^{\frac{1}{2}}};$$ e tangente suo x per integrationem formulæ